PHYS 798C Spring 2024
Lecture 4 Summary

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I. THE MACROSCOPIC QUANTUM MODEL OF SUPERCONDUCTIVITY

The key statement is the following: Superconductivity is inherently a quantum mechanical phenomenon that manifests itself on macroscopic scales. Given this, we now develop a macroscopic quantum model to "explain" superconducting phenomena, especially those associated with the third hallmark of superconductivity. Note that we do this before developing the *microscopic* theory of superconductivity. However, many of the results derived here also hold up for the microscopic case.

A. Review of relevant concepts from Quantum Mechanics

Review of Basic Quantum (wave) Mechanics for *single* particles:
Time-dependent Schrodinger equation: $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{r})\psi$
Probability amplitude for finding the particle: $P(\vec{r},t) := \psi^*(\vec{r},t)\psi(\vec{r},t)$
Normalization condition on the wavefunction: $\int P(\vec{r},t)dV = \int \psi^*(\vec{r},t)\psi(\vec{r},t)dV = 1$ for all time t.
Probability current: $\vec{J}_{prob} = \frac{\hbar}{i2m}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*) = Re[\psi^*\frac{\hbar}{im}\vec{\nabla}\psi]$. Note that $\vec{J}_{prob}$ has dimensions of inverse time.
Continuity equation for probability density: $\frac{\partial P}{\partial t} = -\vec{\nabla}\cdot\vec{J}_{prob}$
Charged particle under the influence of electric and magnetic fields, with associated scalar and vector potentials: $\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$. The canonical momentum is the sum of the kinematic momentum and electromagnetic momentum: $m\vec{v} + q\vec{A}$.
Schrodinger equation including ϕ and $\vec{A}$: $i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}(\frac{\hbar}{i}\vec{\nabla} - q\vec{A})^2\psi + q\phi\psi$
Probability current including electromagnetic momentum $q\vec{A}$: $\vec{J}_{prob} = Re[\psi^*(\frac{\hbar}{im}\vec{\nabla} - \frac{q}{m}\vec{A})\psi]$.

B. Macroscopic Quantum Treatment of Superconductors

Hypothesis: There exists a macroscopic quantum wavefunction $\Psi(\vec{r},t)$ that describes the behavior of the entire ensemble of super-electrons in the superconductor.
Here $\Psi(\vec{r},t)$ is a field-like quantity that describes the coherent behavior of the super-electrons.
Normalization constraint for the Macroscopic Quantum Wave Function (MQWF): $\int \Psi^*(\vec{r},t)\Psi(\vec{r},t)dV = N^*$, where N^* is the total number of super-electrons that the MQWF describes. Note that * is NOT complex conjugation here (N is real)!
Therefore, the local density of super-electrons is $\Psi^*(\vec{r},t)\Psi(\vec{r},t) = n^*(\vec{r},t)$. Note that $|\Psi(\vec{r},t)|^2$ is no longer a probability but in fact describes the location of a sub-set of all of the super-electrons.

Thus the flow of probability $\vec{J}_{prob}$ now describes an actual flow of particles, or a true physical current. We can write the super-current density as $\vec{J}_s = q^*\,\mathrm{Re}\left\{\Psi^*\left(\frac{\hbar}{\imath m^*}\vec{\nabla} - \frac{q^*}{m^*}\vec{A}\right)\Psi\right\}$. We take the super-electrons to have charge q^*, mass m^*, and density n^*, all real quantities.

In polar format, we expect the MQWF to be of the form $\Psi(\vec{r},t) = \sqrt{n^*(\vec{r},t)}e^{\imath\theta(\vec{r},t)}$, where $n^* = \Psi^*\Psi$ and $\theta(\vec{r},t)$ is a real phase factor. Putting this version of Ψ in to the current density expression, we find $\vec{J}_s = q^*n^*(\vec{r},t)\left(\frac{\hbar}{m^*}\vec{\nabla}\theta(\vec{r},t) - \frac{q^*}{m^*}\vec{A}(\vec{r},t)\right)$. (What happened to the ∇n^* term? It disappeared when you take the Real part of the expression!)

Or, using $\vec{J}_s(\vec{r},t) = n^*(\vec{r},t)\,q^*\vec{v}_s$, we can write for the super-fluid velocity $\vec{v}_s = \frac{\hbar}{m^*}\vec{\nabla}\theta(\vec{r},t) - \frac{q^*}{m^*}\vec{A}(\vec{r},t)$. Hence the (measurable) superfluid current density is related to the gradient of the phase of the MQWF and the vector potential, neither of which can be directly measured!

The vector potential reproduces the (measurable) magnetic field $\vec{B}$ through its curl $\vec{B} = \vec{\nabla}\times\vec{A}$, but it can be modified by the gradient of any real scalar function of position and produce the same magnetic field: $\vec{A}\to\vec{A}' = \vec{A} + \vec{\nabla}\chi$. This flexibility in gauge choice also constrains the MQWF phase through $\theta\to\theta' = \theta + \frac{q^*}{\hbar}\chi$. With this change of gauge one can show that the supercurrent density $\vec{J}_s(\vec{r},t)$ is gauge invariant.

C. Generalized London relation

Taking $m^* = 2m$, $q^* = -2e$ and $n^* = n/2$ one can see that $\Lambda^* = \Lambda$! This allows us to write the generalized London relation as follows
$\Lambda \vec{J}_s = \frac{\hbar}{q^*} \vec{\nabla}\theta - \vec{A}$.
Taking the curl of both sides gives the second London equation. Note that the "quantum mechanics" drops out when the curl is taken!
Taking the time derivative of both sides of the London relation gives the first London equation once the time-derivative of the phase of the MQWF is interpreted as an energy (see Homework 2) and the gradient gives the electric field derived from the electric potential ϕ, with the gauge change $\phi \to \phi' = \phi - \partial\chi/\partial t$.

D. Fluxoid Quantization

Consider a closed contour C that is *entirely within a superconductor*. Integrate the generalized London relation around this contour:
$\oint_C \left(\Lambda \vec{J}_s\right) \cdot d\vec{l} = \frac{\hbar}{q^*} \oint_C \vec{\nabla}\theta \cdot d\vec{l} - \oint_C \vec{A} \cdot d\vec{l}$
We can use Stoke's theorem on the last term (only). This last term yields the magnetic flux Φ through any surface S that terminates on the contour C: $\oint_C \vec{A} \cdot d\vec{l} = \iint_S \vec{\nabla} \times \vec{A} \cdot d\vec{S} = \iint_S \vec{B} \cdot d\vec{S} = \Phi_S$.
Why not apply Stoke's theorem to the other two terms? Because the MQWF and $\vec{J}_s$ are not defined outside of the superconductor! Hence it makes no sense to look at the flux of these quantities through surfaces that are outside of the superconductor. (In contrast, note that the vector potential is well-defined everywhere in space.)
The middle term is the integral of the gradient of the phase of the MQWF. With careful analysis noting the 2π ambiguity of the phase, one finds that the integral becomes: $\frac{\hbar}{q^*} \oint_C \vec{\nabla}\theta \cdot d\vec{l} = \frac{\hbar}{q^*} 2\pi p$, where p can be any positive or negative integer, or zero.
Now we have: $\oint_C \left(\Lambda \vec{J}_s\right) \cdot d\vec{l} + \iint_S \vec{B} \cdot d\vec{S} = \frac{h}{q^*} p$. This is a statement of "**fluxoid quantization**". The left hand side of the equation is the fluxoid, and the right hand side is a special combination of fundamental constants known as the flux quantum, $\Phi_0 = h/2e$ where h is Planck's constant and e is the electronic charge. The factor of 2 was put in by hand here, but it is the value seen in experiments on trapped flux in superconductors. We will see later that it comes from the phenomenon of Cooper pairing of the electrons.

Note that only in the case where the contour C is chosen in such a way that the current contour integral is zero do you have the special case of "flux quantization," $\Phi = p\Phi_0$. One way to do this is to have a multiply connected superconductor (e.g. donut or bagel-shaped) in which C is chosen deep inside the superconductor such that $J_s = 0$ there. Then the flux through any surface S that terminates on C will be quantized in units of Φ_0.

The class web site shows data for the trapped flux in a superconducting cylindrical donut as a function of applied magnetic field. The discrete steps in magnetic moment of the trapped flux is a clear and unambiguous sign of flux quantization.

How does a superconducting loop maintain a quantized value of magnetic flux when it is subjected to an arbitrary amount of classical flux? The answer is that it adjusts the screening currents circulating in the loop to keep the overall flux quantized. The total flux is the sum of the externally-applied flux and the 'self flux' created by the superconducting loop: $\Phi_{Total} = \Phi_{applied} - LI$, where L is the self-inductance of the loop and I is the superconducting circulating current. It is this total flux that is quantized in units of Φ_0. This is a 'circuit version' of fluxoid quantization. It is used frequently in the literature, but I find it rather sloppy. It is better to use the fields and currents version of the fluxoid quantization condition, in my opinion.

The final Feynman lecture on Physics was a seminar on the macroscopic quantum model of superconductivity and flux quantization, among other things. It has some very interesting insights about quantum mechanics and superconductivity.

One other note about fluxoid quantization. The argument sketched above is based on the description of the macroscopic quantum wavefunction in terms of a complex function of space and time. This is certainly valid for most superconductors. However, there exist some superconductors (as well as ^3He) that are described by more complicated order parameters, including vector or tensor quantities, rather than just a complex scalar function. Fluxoid quantization is not satisfied, in general, for these types of superconductors. For further discussion, see James Annett, *Superconductivity, Superfluids and Condensates*, Oxford University Press, 2004, p. 158.

E. The Josephson Effects

The macroscopic quantum wavefunction description of a superconductor allows us to understand the starting point for the Josephson effects. When two superconductors (described by MQWFs $\Psi_1 = |\Psi_1|\, e^{i\theta_1}$ and $\Psi_2 = |\Psi_2|\, e^{i\theta_2}$) are brought close together, but spearated by a finite thickness non-superconducting barrier, there can be tunneling of Cooper pairs of electrons between the two materials. Cooper pairing will be discussed in the next lecture. The direction and magnitude of the tunneling supercurrent is given by the DC Josephson equation: $I = I_c \sin\delta$, where the gauge-invariant phase difference is given by $\delta = \theta_1 - \theta_2 - \frac{2e}{\hbar}\int_1^2 \vec{A}\cdot d\vec{l}$, where $\vec{A}$ is the vector potential inside the barrier and the integral is carried out from one superconductor to the other. The gauge-invariant phase difference involves two quantities that cannot be directly measured (θ and $\vec{A}$), but they conspire to construct a quantity that *can* be measured. The DC Josephson equation says that a spontaneous supercurrent flows between the superconductors in the absence of a potential difference, and the sign of the current depends on the realtive phases of the macroscopic wavefunctions and the vector potential in the barrier.

If one establishes a DC potential difference V between the superconductors (because there is a non-superconducitng region between them that supports the potential difference), the gauge-invariant phase difference will evolve linearly as a function of time, as described by the AC Josephson equation: $\frac{d\delta}{dt} = \frac{2e}{\hbar}V$. The magnitude of the potential difference dictates the rate at which $\delta(t)$ evolves. When $\delta(t)$ is substituted back into the DC Josephson equation one sees that it produces an AC supercurrent in the junction! The rate at which the current oscillates is directly proportional to the voltage V. This make the Josephson junction a voltage-controlled oscillator.